CMPLXSYS/POLSCI 391 — Winter 2022 — Discussion
Markov models
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Group names: ______________________________

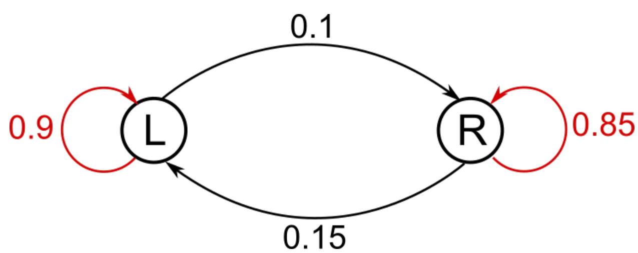
Legend

Text in red are questions that need to be submitted as a part of the in-class discussion (group work, just hand over your sheets).

Text in blue are questions that need to be submitted as a part of the homework due before next lab (upload individual answers on canvas).

Two State Markov System

Consider the following system consisting of 100 people in a population that can either be left-leaning (L) or right-leaning (R). Initially (time=0), the population consists of **50** right-leaning and **50** left-leaning individuals. Assume that L individuals switch their political ideology to R with a 10% probability at every time step. Instead, R individuals switch to L with a probability of 15%. This can be represented as a Markov chain diagram,



1. Given these probabilities (0.1 and 0.15), how did I get the numbers in red in the above diagram? (Explain the concept and not the mathematical rule)

2. What is the number of individuals that transition from L to R and from R to L at time zero? (can be fractional)

- $n(L \to R)_0 =$
- $n(R \to L)_0 =$

3. What is the total number of people with L and R political beliefs at time one? (Hint: Subtract all the people that transition $L \to R$ from the current population of L and add them to R, repeat this for $R \to L$ transition)

- $n(L)_1 =$

- $n(R)_1 =$

From these example steps, you can guess that,

$$n(L)_t = n(L)_{t-1} - n(L \rightarrow R)_{t-1} + n(R \rightarrow L)_{t-1} \tag{1}$$

and similarly,

$$n(R)_t = n(R)_{t-1} - n(R \rightarrow L)_{t-1} + n(L \rightarrow R)_{t-1} \tag{2}$$

4. Write equation (1) and (2) in your own words.

5. Repeat the above process up to time step 15 and fill in the table below.

t	$n(L)$	$n(R)$	$n(R \rightarrow L)$	$n(L \rightarrow R)$
0				
1				
2				
3				
4				
5				
6				
7				
8				
9				
10				

6. What happens to the values of $n(L \rightarrow R)$ and $n(R \rightarrow L)$ as we move further in time. (compare)

7. Calculate the ratio of $n(L)$ and $n(R)$ at the end. Compare it to the ratio of the transition rates (black arrows in the figure).

(Read this after completing the above exercises) You will find the following interesting things from the above calculations,

1. The values of $n(L)$ and $n(R)$ approach an "equilibrium" as we move further in time.
2. The number of people switching at every time-step becomes equal for both transitions at equilibrium.
3. The equilibrium ratio of people in L and R (i.e., $n(L)/n(R)$) is exactly equal to the ratio of transition rates $n(R \to L)/n(L \to R)$.

Thus the equilibrium of a Markov processes doesn't depend on the initial conditions but only on the different transition rates present in the system. You can also verify this by starting with a different number of people in both states at the start.

Multi-State Markov Systems

In general, a Markov system can have any number of states with multiple transition probability terms. We can either use a diagram (like the one in the first question) or a transition matrix to represent such a process.

Properties of a transition matrix

1. It contains an equal number of rows and columns, this number being equal to the total number of states.
2. The number in the i^{th} column and the j^{th} row is the probability of moving from $i \to j$.

8. Create a transition matrix for the diagram in the first part.

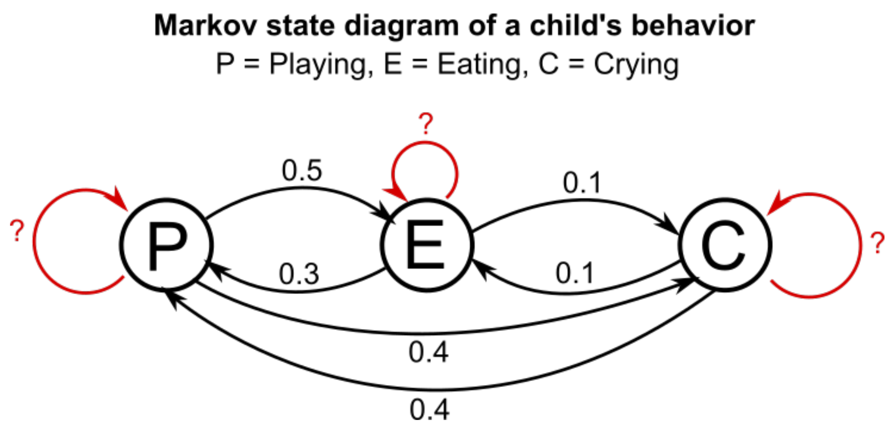
Thanks to decades of development in the mathematical theory of random processes, the transition matrix can be used to solve for the equilibrium of such a process really easily. The process is called finding an"eigenvector". For all the questions below you can use the online eigenvector calculator here: `https://www.arndt-bruenner.de/mathe/scripts/engl_eigenwert2.htm`

9. Find the eigenvector (with the highest eigenvalue) for the transition matrix in question 8.

10. What is the ratio of the two values in the eigenvector. Compare it to the equilibrium ratio of $n(L)$ and $n(R)$ we got in question 7.

Homework

1. Write down the transition matrix for the following markov process (Note that the self-edges in red are missing, you'll have to calculate them yourself; see question 1 for help)



2. Find the principal eigenvector for this process. Based on this answer, what is the proportion of time the child spends playing, eating, and crying?

Discussion Feedback (anonymous; though you'll need to login to your UM google account to prevent unauthorized access): `https://forms.gle/WyP7o66k6hnozcHA8`