CMPLXSYS/POLSCI 391 — Winter 2022 — Discussion
Entropy and MaxEnt
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Question 1: Entropy as uncertainty

1. I have given you a single fair dice that can show one of six faces with equal probabilities (i.e, $p_1 = 1/6, p_2 = 1/6, ...$ and so on). What is the entropy of the probability distribution characterizing the number that shows up on the dice? Use the formula,

$$H(p_1, p_2, ..., p_6) = -\sum_{i=1}^{6} p_i \log_2 p_i \tag{1}$$

Now, I give you another dice that never shows the faces labelled 2 and 4 and is equally likely to show any of the other faces. The probability distribution can be written as,

$$p_1 = 1/4,\ p_2 = 0,\ p_3 = 1/4,\ p_4 = 0,\ p_5 = 1/4,\ p_6 = 1/4$$

2. What is the entropy of the probability distribution now?

3. What happens to the entropy when the number of possible outcomes decreases (i.e., you switch from the first dice to the second one)? (Does it increase or decrease)

4. What about your own feeling of uncertainty about the number that will appear on the top of the dice? Does it increase or decrease when you switch from the first dice to the second one?

Note how entropy, in a way, characterizes the uncertainty you have about the outcome.

Question 2: Change in entropy and information

You are a spy charged with tracking down an asset (a person aiding in the intelligence service but unaware of it) in the city. You know that this person visits one out of - a pub, a cafe, a friend's house, an AA meeting, or a government office - with the following probabilities.

$$\begin{aligned} P(\text{Pub}) &= 1/5 \\ P(\text{Cafe}) &= 1/5 \\ P(\text{Friend's house}) &= 1/10 \\ P(\text{AA Meeting}) &= 1/10 \\ P(\text{Govt. office}) &= 2/5 \end{aligned}$$

An informant tells you that the asset had a fight with their friend and will not be going to their house this evening. The new probabilities are,

$$\begin{aligned} P(\text{Pub}) &= 2/9 \\ P(\text{Cafe}) &= 2/9 \\ P(\text{Friend's house}) &= 0 \\ P(\text{AA Meeting}) &= 1/9 \\ P(\text{Govt. office}) &= 4/9 \end{aligned}$$

1. Calculate the entropy of the asset's probability distribution among the different places for both cases (before and after receiving the information).

2. Let us measure the information received as a difference in uncertainties before and after the informant's message. What is the amount of information received from the informant? Use the following formula for this calculation.

$$\text{Information} = H(\text{before}) - H(\text{after})$$

3. How much information would you have received if the informant had told you exactly where the asset was going to be in the evening?

Question 3: Maximum entropy estimates (MaxEnt)

The principle of maximum entropy states that the most unbiased and provisionally accurate probability distribution that characterises a system is the one that maximizes the entropy given current information. Consider the following case,

1. Once again, you are given a dice (but an unfair one). You are told that the number 1 appears with a probability 1/2 (unlike 1/6 in the case of a fair dice). What are the probabilities with which other numbers on the die must appear for you to have the maximum uncertainty about the outcome? (You do not need to calculate entropies in this case, just think about what the probabilities should be for you to be the most uncertain, also remember that the probabilities for all faces sum up to 1)

2. Calculate the entropy of the distribution with these probabilities that maximize uncertainty.

3. Change the probabilities (of faces other than 1) slightly and recalculate the entropy. Is it higher or lower than the entropy in the previous question?

In the future, whenever you're trying to estimate something given some information, you can use the MaxEnt method to make sure that your estimates are not biased and are only as certain as allowed by the information you already have!

An optional but highly recommended article to learn more about the concepts of uncertainty, entropy, and information: `https://doi.org/10.1098/rsta.2015.0230`

Discussion Feedback (anonymous; though you'll need to login to your UM google account to prevent unauthorized access): `https://forms.gle/WyP7o66k6hnozcHA8`